

INTERSECTIONS OF COMBINATORIAL BALLS AND OF EUCLIDEAN SPACES

BY
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1. Introduction. Poenaru [5] and Mazur [4] gave the first examples of contractible compact combinatorial 4-manifolds with boundary which were not topological 4-cells, but whose products with the unit interval were combinatorial 5-cells. Curtis [1] and Glaser [3] gave similar examples for $n \geq 5$. In the latter result the product of the pseudo n -cell M^n with an interval was shown to be a combinatorial $(n + 1)$ -cell rather than just merely topological. In addition, it was shown in [3], that for $n \geq 5$, M^n was a compact combinatorial n -manifold with boundary not topologically I^n , but could be expressed as the union of two combinatorial n -cells whose intersection is also a combinatorial n -cell. Unfortunately the techniques used in [3] gave no hope of lowering the result to $n = 4$.

The purpose of this paper is to give another example of a pseudo 4-cell W with the property that $W \times I \approx I^5$, but in addition W also can be expressed as the union of two combinatorial 4-cells whose intersection is also a combinatorial 4-cell. This also gives an example of two Euclidean 4-spaces intersecting in an Euclidean 4-space so that the union is not topologically E^4 .

2. Definitions. We will use the standard terminology I^n , E^n , and S^n for the unit n -cell, Euclidean n -space and the n -sphere respectively. If M is an n -manifold, then $\text{int } M$ and $\text{Bd } M$ will denote the interior and boundary of M , respectively. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. Topological equivalence will be denoted by $=$, and we will use $\approx$ to denote combinatorial equivalence. We will use technique of collapsing polyhedra, denoted by $\searrow$, and the notion of regular neighborhoods as in Whitehead [7] or Zeeman [8].

3. Construction. In this section we will give an example of a certain contractible 2-complex K and an embedding of K in a combinatorial 4-manifold W with boundary so that $\pi_1(\text{Bd } W) \neq 1$ and W can be considered as a regular neighborhood of K . W will be the pseudo 4-cell promised in the introduction.

K is obtained by attaching two disks along a figure eight. Let us consider the figure eight as four line segments α, β, γ and δ and three vertices a, b , and c as indicated in Figure 1. The two disks are attached by the formula $\beta\gamma\gamma^{-1}\delta^{-1}\delta\alpha$

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and $\delta\alpha\alpha^{-1}\beta^{-1}\beta\gamma$. The resulting 2-complex K is also indicated in Figure 1. We observe that K is a contractible noncollapsible 2-complex by noting that we can easily get K as a deformation retract of a 3-cell and that K has no free edges.

Let T be a solid two-holed 3-dimensional torus in E^3 . Let us consider two simple closed curves Γ_1 and Γ_2 embedded in $\text{int}(T \times \{1\}) \subset T \times [0, 1]$ as indicated in Figure 2. W will be formed by attaching two 2-handles to the boundary of $T \times [0, 1]$ along the curves Γ_1 and Γ_2 .

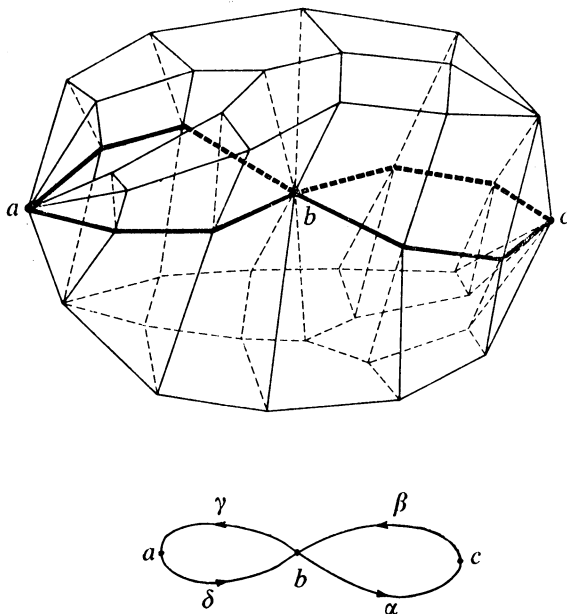


FIGURE 1

More precisely, let j be an embedding of $\text{Bd } I^2 \times I^2 \rightarrow \text{int}(T \times \{1\})$ such that $j(\text{Bd } I^2 \times 0) = \Gamma_1$ and k an embedding of $\text{Bd } I^2 \times I^2 \rightarrow \text{int}(T \times \{1\}) - j(\text{Bd } I^2 \times I^2)$ such that $k(\text{Bd } I^2 \times 0) = \Gamma_2$, where $0 \in \text{int } I^2$. Also let us choose j and k , so that in forming the tubular neighborhoods $j(\text{Bd } I^2 \times I^2)$ and $k(\text{Bd } I^2 \times I^2)$ we do not have any twisting as we go around each of Γ_1 and Γ_2 respectively.

Define W as $I^2 \times I^2 \cup T \times [0, 1] \cup_k I^2 \times I^2$.

LEMMA 1. *W can be considered as a regular neighborhood of a combinatorial embedding of K in W .*

Proof. Divide $T \times \{1\}$ into seven 3-cells $B_1, B_2, \dots, B_7$ as indicated in Figure 2. Let us denote the figure eight forming the core of $T \times \{\frac{1}{2}\}$ by α, β, γ and δ as we did in defining K . This is also indicated in Figure 2. Let us triangulate $\text{Bd}(T \times [0, 1]) \approx 2T$ so that $\Gamma_1, \Gamma_2, j(\text{Bd } I^2 \times I^2), k(\text{Bd } I^2 \times I^2), B_1, B_2, \dots, B_7$ are subcomplexes of our triangulation. We also triangulate each copy of $I^2 \times I^2$ so that $j^{-1}(j(\text{Bd } I^2 \times I^2))$ and $k^{-1}(k(\text{Bd } I^2 \times I^2))$ are subcomplexes of their respective 4-cells. Next we want to extend the triangulation of $\text{Bd}(T \times [0, 1])$

which we now will consider as $2T$ to $T \times [0, 1]$ so that the figure eight $\alpha\beta\gamma\delta$ is a subcomplex of $T \times [0, 1]$, $K \subset T \times [0, 1]$ and so that $W \searrow K$.

In considering $\text{Bd}(T \times [0, 1])$ as $2T$ let $B'_1, B'_2, \dots, B'_7$ denote the corresponding 3-cells of the other copy of T . Now we triangulate $T \times [0, 1]$ so that the cones $a(B_1 \cup B'_1)$, $b(B_4 \cup B'_4)$ and $c(B_7 \cup B'_7)$ are subcomplexes of $T \times [0, 1]$. Let us denote these cones as C_1 , C_4 and C_7 respectively. Each of $B_2 \cup B'_2$, $B_3 \cup B'_3$, $B_5 \cup B'_5$ and $B_6 \cup B'_6$ can be considered as a copy of $[0, 1] \times S^2$. For notational purposes we will denote this as $[0, 1]_i \times S^2$, $i = 2, 3, 5, 6$. Let f_2 be a simplicial homeomorphism taking $[0, 1]_2$ onto γ ; similarly, $f_3: [0, 1]_3 \rightarrow \delta$, $f_5: [0, 1]_5 \rightarrow \beta$

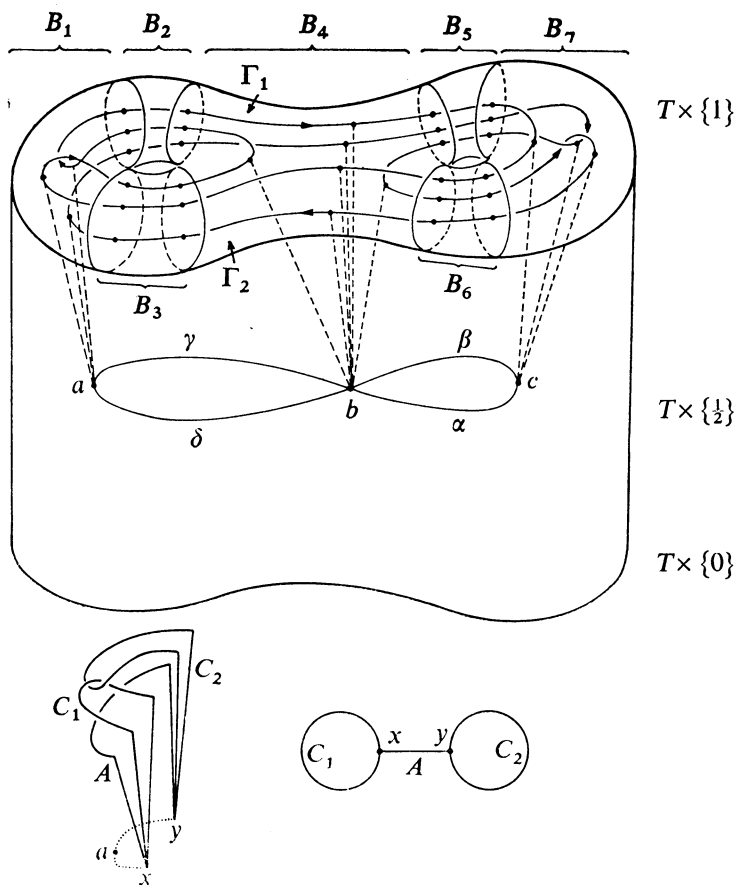


FIGURE 2

and $f_6: [0, 1]_6 \rightarrow \alpha$. Let g_i be the simplicial map taking $[0, 1]_i \times S^2$ onto the appropriate segment by taking $[0, 1]_i \times S^2 \rightarrow [0, 1]_i$ and then following this by f_i . Let M_i denote the mapping cylinders of g_i , $i = 2, 3, 5, 6$. Now map each $[0, 1]_i \times S^2 \subset M_i$ homeomorphically onto $[0, 1]_i \times S^2 \subset 2T$. Next extend the map so that $a(\{0\}_2 \times S^2)$, $a(\{0\}_3 \times S^2)$, $b(\{1\}_2 \times S^2)$, $b(\{1\}_3 \times S^2)$, $b(\{0\}_5 \times S^2)$,

$$\begin{aligned}
\text{IV. } & \bar{y}(\bar{b} y z b \bar{z} \bar{y} b y z \bar{b} \bar{z} \bar{y} b) (\bar{b} \bar{a} y a x b \bar{z}) (\bar{b} y z b \bar{z} \bar{y} \bar{b} y z \bar{b} \bar{z} \bar{y} b) = 1, \\
\Gamma_1: & \bar{x}(\bar{a} \bar{y} a) (\bar{a} \bar{x} \bar{a} \bar{y} \bar{a} y a x a) x \bar{a} = 1, \\
\Gamma_2: & \bar{z} \bar{y} b y (\bar{b} y z b \bar{z} \bar{y} b) = 1, \\
K_1: & (x \bar{a} \bar{x} \bar{a} \bar{y} a y a x a \bar{x}) \bar{a} \bar{b} = 1, \\
K_2: & \bar{a}(\bar{b} y z b \bar{z} \bar{y} b y z \bar{b} \bar{z} \bar{y} b) \bar{b} = 1.
\end{aligned}$$

We note that relations I-IV give $\pi_1(E^3 - (\Gamma_1 + \Gamma_2 + K_1 + K_2))$, adding in relations K_1 and K_2 give $\pi_1(2T - (\Gamma_1 + \Gamma_2))$, and adding in relations Γ_1 and Γ_2 gives $\pi_1(\text{Bd } W)$.

Now Γ_1 gives that $\bar{a} \bar{x} \bar{a} \bar{y} \bar{a} y a x a = \bar{a} y a x a \bar{x}$. This relation applied to I gives $1 = 1$. By K_1 we have $x \bar{a} \bar{x} \bar{a} \bar{y} \bar{a} y a x a \bar{x} = b a$. Applying this to II gives that $\bar{x} b z \bar{b} = 1$ or $z = \bar{b} x b$. Using Γ_2 and K_2 in III we get $\bar{y}(\bar{z} \bar{y} b y) y a b = 1$. Using K_2 and the fact that $z = \bar{b} x b$ in IV we get $1 = 1$.

Using Γ_2 , $\bar{z} \bar{y} b y = \bar{b} y z \bar{b} \bar{z} \bar{y} b$ in K_2 gives that $\bar{a}(\bar{y} \bar{b} y z) y z \bar{b} \bar{z} \bar{y} = 1$. Next applying the new relation III $\bar{y} \bar{b} y z = y a b \bar{y}$ and $z = \bar{b} x b$ to the preceding relation for K_2 we then get $x \bar{b} \bar{x} b \bar{y} = \bar{a} \bar{y} a$.

Writing Γ_2 as $\bar{z} \bar{y} b y = \bar{b} y z \bar{b} \bar{z} \bar{y} b$, replacing z by $\bar{b} x b$ and using the fact that $x \bar{b} \bar{x} b \bar{y} = \bar{a} \bar{y} a$ we get that $\bar{x} b \bar{y} b y = y \bar{b} \bar{a} \bar{y} a b$.

In considering III, $\bar{z} \bar{y} b y = y \bar{b} \bar{a} \bar{y}$, if we replace z by $\bar{b} x b$ and $\bar{x} b \bar{y}$ by $b \bar{x} \bar{a} \bar{y} a$, we get $\bar{x} \bar{a} \bar{y} a b y = y \bar{b} \bar{a} \bar{y}$.

Finally, using Γ_1 , $\bar{a} \bar{x} \bar{a} \bar{y} \bar{a} y a x a = \bar{a} y a x a \bar{x}$ in K_1 , gives that $x^2 \bar{a} \bar{x} \bar{a} \bar{y} a \bar{x} \bar{a} = b$.

Our group now has the following presentation:

$$\begin{aligned}
\text{I. } & 1 = 1, \\
\text{II. } & z = \bar{b} x b, \\
\text{III. } & \bar{x} \bar{a} \bar{y} a b y = y \bar{b} \bar{a} \bar{y}, \\
\text{IV. } & 1 = 1, \\
\Gamma_1: & \bar{a} \bar{x} \bar{a} \bar{y} \bar{a} y a x a = \bar{a} y a x a \bar{x}, \\
\Gamma_2: & \bar{x} b \bar{y} b y = y \bar{b} \bar{a} \bar{y} a b, \\
K_1: & x^2 \bar{a} \bar{x} \bar{a} \bar{y} a \bar{x} \bar{a} = b, \\
K_2: & x \bar{b} \bar{x} b \bar{y} = \bar{a} \bar{y} a.
\end{aligned}$$

If we replace the first x in K_2 by using relation III, and replace the $\bar{x}$ by using Γ_2 we get that $a b \bar{y} \bar{b} y \bar{b} \bar{a} \bar{y} a b y = 1$. Using the fact that the x from III equals the x from Γ_2 , that is, that $\bar{a} \bar{y} a b y = b \bar{y} b y \bar{b} \bar{a}$, the preceding relation just becomes $1 = 1$.

Now using III, $x = \bar{a} \bar{y} a b y^2 a b \bar{y}$ we get:

$$\begin{aligned}
\text{III} = \Gamma_2: & \bar{a} \bar{y} a b y = b \bar{y} b y \bar{b} \bar{a}, \\
K_1: & [\bar{a} \bar{y} a (b y^2 a b \bar{y})]^2 \bar{a} (y \bar{b} \bar{a} \bar{y}^2 \bar{b})^2 \bar{a} y = b, \\
\Gamma_1: & \bar{a} (y \bar{b} \bar{a} \bar{y}^2 \bar{b}) \bar{a} (b y^2 a b \bar{y}) = (b y^2 a b \bar{y}) a (y \bar{b} \bar{a} \bar{y}^2 \bar{b}) \bar{a} y.
\end{aligned}$$

Here we have used in K_1 and Γ_1 the fact that III also gives $\bar{a} \bar{y} a = x y \bar{b} \bar{a} \bar{y}^2 \bar{b}$.

Setting $\bar{a}y = \beta$ and $\alpha = y\bar{b}\bar{a}\bar{y}^2\bar{b}$ we get:

$$\Gamma_1: \bar{a}\alpha\bar{a}\bar{\alpha} = \bar{\alpha}a\alpha\beta,$$

$$K_1: (\bar{a}\bar{\beta}\alpha)^2\bar{a}(\alpha^2)\beta = b.$$

Also $x = \bar{a}\bar{\beta}\bar{\alpha}$.

Using Γ_1 to solve for β and applying this to K_1 we have $b = \bar{a}\alpha a\bar{\alpha}a\alpha\bar{a}\bar{\alpha}$. We now have b and β in terms of a and α ; and hence y in terms of a and α also. Thus we now have only two relations to consider. Namely, $\alpha = y\bar{b}\bar{a}\bar{y}^2\bar{b}$ and $\bar{a}\bar{y}a by = b\bar{y}by\bar{b}\bar{a}$. Writing y and b in terms of a and α we get the following group presentation:

generators: a, α ,

relations:

$$\begin{aligned} \bar{\alpha}a\bar{\alpha}\bar{a}\bar{\alpha}\bar{a}\bar{\alpha}a\alpha\bar{a}\alpha a\bar{\alpha} &= \bar{a}\alpha a\bar{\alpha}a\alpha\bar{a}\bar{\alpha}a\bar{\alpha}\bar{a}\alpha\bar{a} \\ &= a\bar{\alpha}\bar{a}\alpha\bar{a}\bar{\alpha}a^2\bar{\alpha}\bar{a}\alpha\bar{a}\bar{\alpha}a\alpha\bar{a}\alpha a\bar{\alpha}a. \end{aligned}$$

Now if we add the relation that $\bar{\alpha}a = \bar{a}\alpha$, the first equality becomes $\alpha^5 = a^5$ and the second $\bar{a}\alpha^3\bar{a}^2\bar{\alpha} = a\bar{\alpha}^3a^2\bar{\alpha}^3a^2\alpha$. Adding the relation $\alpha^5 = a^5 = (a^2\alpha^2)^2 = 1$, we get the group

$$\{a, \alpha \mid \bar{\alpha}a = \bar{a}\alpha, \alpha^5 = a^5 = (a^2\alpha^2)^2 = 1\}.$$

Replacing a^2 by u and α^2 by v we get

$$\{u, v \mid v^2u^3v^2 = u^2, u^5 = v^5 = (uv)^2 = 1\}.$$

This group can be shown to have a nontrivial representation in P_5 by letting $u \rightarrow (12345)$ and $v \rightarrow (12354)$. If we desire to check that this does indeed give a nontrivial representation of the original group, we have the following:

$$\alpha \rightarrow (15243), \quad \beta \rightarrow (254), \quad a \rightarrow (14253), \quad b \rightarrow (12543),$$

$$x \rightarrow (14352), \quad y \rightarrow (12453) \quad \text{and} \quad z \rightarrow (14523).$$

4. Main results. In this section we will discuss some additional properties of the pseudo 4-cell W and show how the particular chosen 2-complex K leads to the desired results.

LEMMA 2. Suppose K is a contractible -subcomplex in the interior of a combinatorial 4-manifold M and W is a regular neighborhood of K in M . If K can be combinatorially embedded in E^3 , then W can be embedded in E^4 and $W \times I \approx I^5$.

Proof. By [1, Proposition 2] $W \times I^2 = I^6$. Since $\text{Bd}(W \times I^2)$ is homeomorphic to S^5 triangulated as a combinatorial 5-manifold and $2(W \times I) \approx \text{Bd}(W \times I^2)$, $W \times I$ can be combinatorially embedded in a combinatorial triangulation of

E^5 . Let K' be a combinatorial embedding of K in $E^3 \subset E^5$. Since the regular neighborhood of K' in E^3 is necessarily a combinatorial 3-cell, the regular neighborhood N of K' in E^5 is a combinatorial 5-cell. By the corollary of [6], this implies that $W \times I \approx N \approx I^5$. The fact that $2W \approx \text{Bd}(W \times I) \approx S^4$ gives that W can be combinatorially embedded in E^4 .

THEOREM 2. *There exists a pseudo 4-cell $W \neq I^4$ such that $W \subset E^4$, $W \times I \approx I^5$ and $W \approx X \cup Y$, where $X \approx Y \approx X \cap Y \approx I^4$.*

Proof. W is the pseudo 4-cell of §3. Since $\pi_1(\text{Bd } W) \neq 1$ we have $W \neq I^4$. Since $W \searrow K$ and K can be embedded in E^3 , the fact that $W \subset E^4$ and $W \times I \approx I^5$ follows from Lemma 2.

Let A be the middle polyhedral arc going from the vertex b to the vertex c in the top disk used in the construction of K . Similarly, let B be the middle polyhedral arc going from the vertex a to the vertex b in the bottom disk. If we separate K along the polyhedral arc $B \cup A$ we end up with two collapsible complexes which we will denote as K_1 and K_2 . Hence $K \equiv K_1 \cup K_2$, $K_1 \cap K_2 \equiv B \cup A$ and each of K_1 , K_2 , and $K_1 \cap K_2$ collapses to a point. Let W' be a regular neighborhood of K in W under the secondary centric subdivision of W . Let X' be the regular neighborhood of K_1 and Y' the regular neighborhood of K_2 under this subdivision. Now $X' \cap Y'$ is combinatorially equivalent to the regular neighborhood of $K_1 \cap K_2 \equiv B \cup A$. Since $X' \searrow K_1 \searrow 0$, $Y' \searrow K_2 \searrow 0$, and $X' \cap Y' \searrow B \cup A \searrow 0$ we have $X' \approx Y' \approx X' \cap Y' \approx I^4$ by the results of Whitehead [7]. Again using [7] we have that $W \approx W'$ and hence the conclusion to the theorem.

COROLLARY 1. *For $n \geq 4$ there exist pseudo n -cells $W^n \neq I^n$ such that*

$$W^n \times I \approx I^{n+1}$$

and $W^n \approx X^n \cup Y^n$, where $X^n \approx Y^n \approx X^n \cap Y^n \approx I^n$.

Proof. The result for $n = 4$ is just Theorem 2 and for $n \geq 5$ follows from [3].

COROLLARY 2. *For $n \geq 3$ there exists open contractible combinatorial n -manifolds $O^n \neq E^n$ such that $O^n \approx U^n \cup V^n$, where $U^n \approx V^n \approx U^n \cap V^n \approx E^n$.*

Proof. The result for $n \geq 5$ follows from [3]. For $n = 4$ we use $U \approx \text{int } X$, $V \approx \text{int } Y$ and $O^4 \approx \text{int } W$ of Theorem 2. We have that $O^4 \neq E^4$ since $\pi_1(\text{Bd } W) \neq 1$. That is, if $O^4 = E^4$ then simple closed curves near "infinity" could be shrunk near "infinity", but the collar of $\text{Bd } W$ is not simply connected.

For $n = 3$, the result has been known for some time, but apparently is not too well known. Hence for completeness, the example will be included here. Consider the double Fox-Artin arc A in S^3 intersecting the 2-sphere S^2 in the point p as indicated in Figure 4. Taking U' and V' as the two components of $S^3 - S^2$, one can easily express each of $U' - A$ and $V' - A$ as a monotone increasing sequence of open 3-cells. Let C' be a small double collar of S^2 so that $C' \cap A$ is an open

straight line segment and let $C = C' - A$. Then taking $U = (U' - A) \cup C$ and $V = (V' - A) \cup C$ we have $S^3 - A = U \cup V$ where $U \approx V \approx E^3$ and $U \cap V \approx E^3$ since $U \cap V \approx C \approx \{S^2 - p\} \times (-1, 1)$. We get that $S^3 - A \neq E^3$ since $S^3 - (A + B)$ is not simply connected, where B is the simple closed curve indicated in Figure 4. That is, if $S^3 - A = E^3$, then simple closed curve near "infinity" (here this means curves in an arbitrarily small neighborhood of A in S^3) could be shrunk missing B and this will not always be possible.

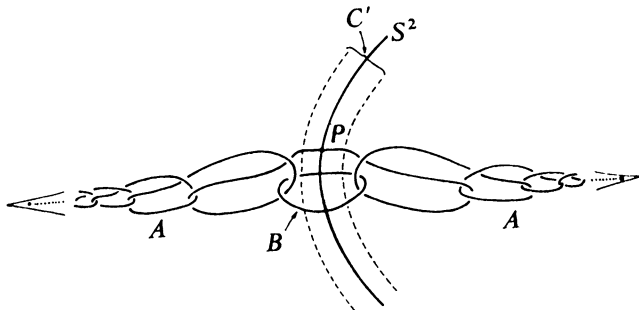


FIGURE 4

Clearly in the construction of W we could have altered slightly our embeddings of Γ_1 and Γ_2 in $\text{int } T$, say link Γ_1 or Γ_2 with itself differently, add local knots, or link Γ_1 with Γ_2 , and still get a contractible 4-manifold with boundary which also collapses to K . Also, it is interesting to note that in some sense the given embeddings are the simplest possible in order to get an example where $\pi_1(\text{Bd } W) \neq 1$. In fact the crucial part of the construction is the linking of Γ_1 over a and the linking of Γ_2 over c . Moreover, our next result says that as long as $\text{lk}(a, K)$ and $\text{lk}(c, K)$ are "nice", no matter how badly Γ_1 and Γ_2 are locally knotted or linked together in the middle section of T , if we repeat the same construction the resulting W is indeed $\approx I^4$.

In the following we apply some of the techniques of [8]. It is easy to see that each of $\text{lk}(a, K)$ and $\text{lk}(c, K)$ is merely two circles, C_1 and C_2 say, joined by an arc $A = xy$ (refer to Figure 2). We will say that the embedding of K in the interior of a combinatorial 4-manifold M^4 is nice at a if $\text{lk}(a, K)$ in $\text{lk}(a, M^4) \approx S^3$ is such that there exist a 2-sphere S^2 in $\text{lk}(a, M^4)$ separating C_1 and C_2 and meeting A in a single point $z \in \text{int } A$. Similarly for the vertex c . We note in the given construction that we have embedded K in W so that the circles corresponding to C_1 and C_2 in each of $\text{lk}(a, K)$ and $\text{lk}(c, K)$ are linked in $\text{lk}(a, W)$ and $\text{lk}(c, W)$ respectively.

THEOREM 3. *Let $K \subset \text{int } M^4$ and suppose $M^4 \searrow K$. If the embedding of K is nice at a and c , then $M^4 \approx I^4$.*

Proof. Let us write $\text{lk}(a, K) = C_1 \cup A \cup C_2$ and $\text{lk}(c, K) = C_1' \cup A' \cup C_2'$. There exists a 2-dimensional polyhedron P such that:

- (i) $C_1 \subset P \subset \text{lk}(a, M^4)$;
- (ii) $P \searrow x$;
- (iii) $P \cap A = x$;
- (iv) $P \cap C_2 = \emptyset$.

Such a P is not difficult to get and the actual construction of such a polyhedron is given in the proof of Theorem 8 of [8]. Similarly there exists a P' such that:

- (i) $C'_1 \subset P' \subset \text{lk}(c, M^4)$;
- (ii) $P' \searrow x'$;
- (iii) $P' \cap A' = x'$;
- (iv) $P' \cap C'_2 = \emptyset$.

Now C_1 intersects either γ or δ in a single point x and C'_1 intersects one of α or β in x' . Recall we used α, β, γ and δ in defining K (refer back to Figure 1). For notational purposes let us suppose that $C_1 \cap \gamma \neq \emptyset$ and $C'_1 \cap \alpha \neq \emptyset$. Now we have the following: $M^4 \searrow K \nearrow K \cup aP \nearrow K \cup aP \cup cP'$. Since $P \searrow x$ and $P' \searrow x'$ we have $aP \searrow ax \cup P$ and $cP' \searrow cx' \cup P'$. Therefore,

$$K \cup aP \cup cP' \searrow \text{Cl}(K - aC_1 - cC'_1) \cup P \cup P'$$

which we will denote by K' .

Let us consider the top half of K' . cx' is now a free edge and hence we can collapse the right half and back part of the top half to the remainder $\cup P$. Then we can collapse $P \searrow x$ and the remaining complex of the top half to δ . Similarly, in considering the bottom half of K' , we have that ax is a free edge on this half and hence we can collapse this half to β . Hence we have $K' \searrow \delta \cup \beta \searrow b$. We now have obtained a sequence of elementary collapses and expansions going from M^4 to b ; hence by Lemma 3 of [8], $M^4 \approx I^4$.

COROLLARY 3. *If $K \subset \text{int } M^n$ ($n \geq 5$) and $M^n \searrow K$, then $M^n \approx I^n$.*

Proof. Since $n \geq 5$ we can get C_1 to bound a disk P in $\text{lk}(a, M^n)$ and C_1 to bound a disk P' in $\text{lk}(c, M^n)$ with the same properties as the P and P' of Theorem 3.

We can also prove Corollary 3 by making use of [6]. That is $M^n \times I = I^{n+1}$ and hence M^n can be embedded in a combinatorial triangulation of E^n . Since K can be embedded in E^3 , say as K' , and $n \geq 5$, the corollary of [6] says that the regular neighborhoods of K' and K in E^n are combinatorially equivalent and hence $M^n \approx I^n$.

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